Spring 2012

EE 330

ENGINEERING ELECTROMAGNETICS

HW 4: *Due Friday 3 February*
2.23, 2.39, 2.50, 2.55, 2.66, 2.68, 2.82,
3.5, 3.15, 3.19, 3.25, 3.36, 3.44, 3.46, 3.52

Problem 2.23 A load with impedance $Z_L = (25 - j50)\ \Omega$ is to be connected to a lossless transmission line with characteristic impedance Z_0, with Z_0 chosen such that the standing-wave ratio is the smallest possible. What should Z_0 be?

Solution: Since S is monotonic with $|\Gamma|$ (i.e., a plot of S vs. $|\Gamma|$ is always increasing), the value of Z_0 which gives the minimum possible S also gives the minimum possible $|\Gamma|$, and, for that matter, the minimum possible $|\Gamma|^2$. A necessary condition for a minimum is that its derivative be equal to zero:

$$0 = \frac{\partial}{\partial Z_0}|\Gamma|^2 = \frac{\partial}{\partial Z_0}\frac{|R_L + jX_L - Z_0|^2}{|R_L + jX_L + Z_0|^2}$$

$$= \frac{\partial}{\partial Z_0}\frac{(R_L - Z_0)^2 + X_L^2}{(R_L + Z_0)^2 + X_L^2} = \frac{4R_L(Z_0^2 - (R_L^2 + X_L^2))}{((R_L + Z_0)^2 + X_L^2)^2}.$$

Therefore, $Z_0^2 = R_L^2 + X_L^2$ or

$$Z_0 = |Z_L| = \sqrt{(25^2 + (-50)^2)} = 55.9\ \Omega.$$

A mathematically precise solution will also demonstrate that this point is a minimum (by calculating the second derivative, for example). Since the endpoints of the range may be local minima or maxima without the derivative being zero there, the endpoints (namely $Z_0 = 0\ \Omega$ and $Z_0 = \infty\ \Omega$) should be checked also.

Problem 2.39 A 75-Ω resistive load is preceded by a $\lambda/4$ section of a 50-Ω lossless line, which itself is preceded by another $\lambda/4$ section of a 100-Ω line. What is the input impedance? Compare your result with that obtained through two successive applications of CD Module 2.5.

Solution: The input impedance of the $\lambda/4$ section of line closest to the load is found from Eq. (2.97):

$$Z_{in} = \frac{Z_0^2}{Z_L} = \frac{50^2}{75} = 33.33\ \Omega.$$

The input impedance of the line section closest to the load can be considered as the load impedance of the next section of the line. By reapplying Eq. (2.97), the next section of $\lambda/4$ line is taken into account:

$$Z_{in} = \frac{Z_0^2}{Z_L} = \frac{100^2}{33.33} = 300\ \Omega.$$

Problem 2.50 Use the Smith chart to determine the input impedance Z_{in} of the two-line configuration shown in Fig. P2.50.

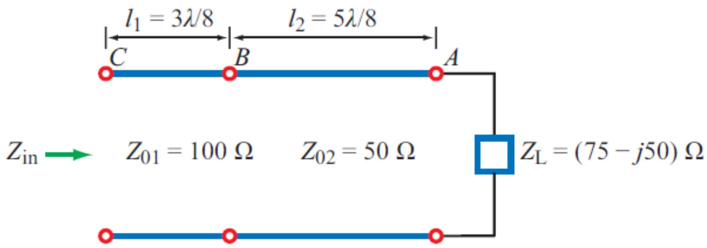
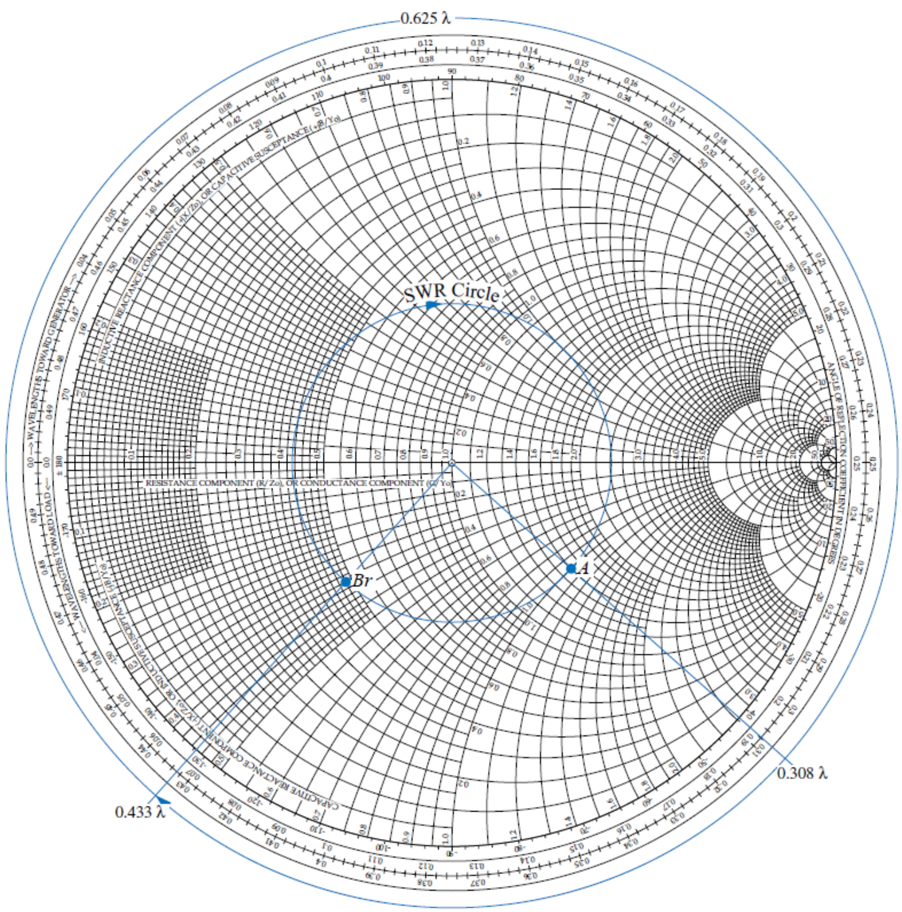


Figure P2.50: Circuit for Problem 2.50.

Solution:



Smith Chart 1

Starting at point A, namely at the load, we normalize Z_{L} with respect to Z_{02}:

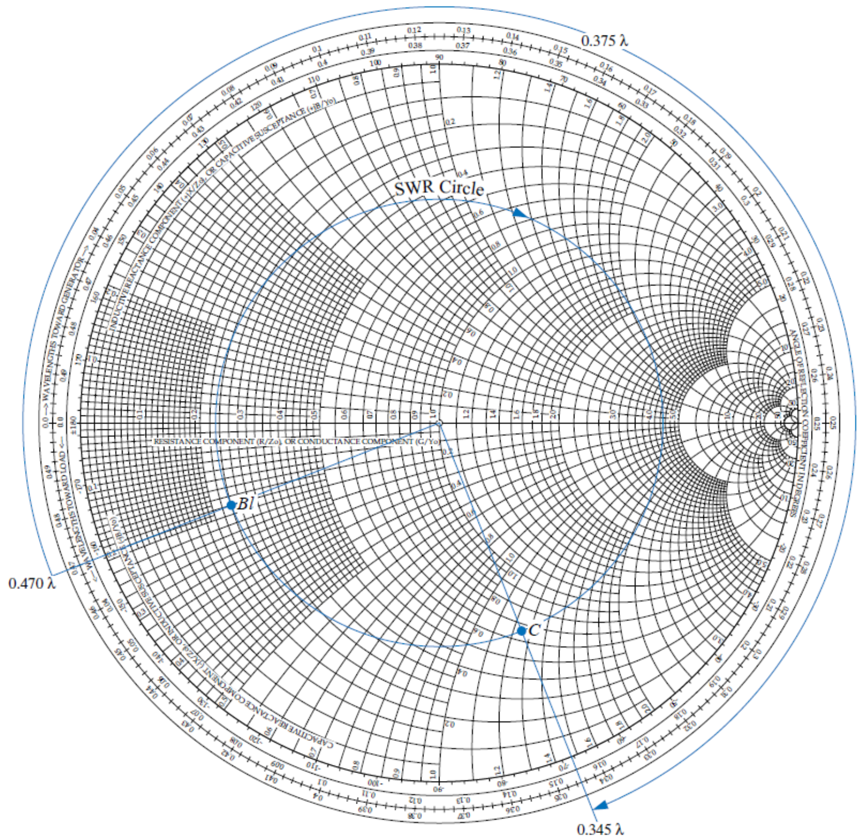
$$z_{\text{L}} = \frac{Z_{\text{L}}}{Z_{02}} = \frac{75 - j50}{50} = 1.5 - j1. \qquad \text{(point } A \text{ on Smith chart 1)}$$

From point A on the Smith chart, we move on the SWR circle a distance of $5\lambda/8$ to point B_{r}, which is just to the right of point B (see figure). At B_{r}, the normalized input impedance of line 2 is:

$$z_{\text{in2}} = 0.48 - j0.36 \qquad \text{(point } B_{\text{r}} \text{ on Smith chart)}$$

Next, we unnormalize z_{in2}:

$$Z_{\text{in2}} = Z_{02} z_{\text{in2}} = 50 \times (0.48 - j0.36) = (24 - j18)\ \Omega.$$



Smith Chart 2

To move along line 1, we need to normalize with respect to Z_{01}. We shall call this z_{L1}:

$$z_{L1} = \frac{Z_{in2}}{Z_{01}} = \frac{24 - j18}{100} = 0.24 - j0.18 \qquad \text{(point } B_\ell \text{ on Smith chart 2)}$$

After drawing the SWR circle through point B_ℓ, we move $3\lambda/8$ towards the generator, ending up at point C on Smith chart 2. The normalized input impedance of line 1 is:

$$z_{in} = 0.66 - j1.25$$

which upon unnormalizing becomes:

$$Z_{in} = (66 - j125)\ \Omega.$$

Problem 2.55 A lossless 50-Ω transmission line is terminated in a short circuit. Use the Smith chart to determine:

(a) The input impedance at a distance 2.3λ from the load.

(b) The distance from the load at which the input admittance is $Y_{in} = -j0.04$ S.

Solution: Refer to Fig. P2.55.

(a) For a short, $z_{in} = 0 + j0$. This is point *Z-SHORT* and is at 0.000λ on the WTG scale. Since a lossless line repeats every $\lambda/2$, traveling 2.3λ toward the generator is equivalent to traveling 0.3λ toward the generator. This point is at A : *Z-IN*, and

$$Z_{in} = z_{in} Z_0 = (0 - j3.08) \times 50\ \Omega = -j154\ \Omega.$$

(b) The admittance of a short is at point *Y-SHORT* and is at 0.250λ on the WTG scale:

$$y_{in} = Y_{in} Z_0 = -j0.04\ \text{S} \times 50\ \Omega = -j2,$$

which is point B : *Y-IN* and is at 0.324λ on the WTG scale. Therefore, the line length is $0.324\lambda - 0.250\lambda = 0.074\lambda$. Any integer half wavelengths farther is also valid.

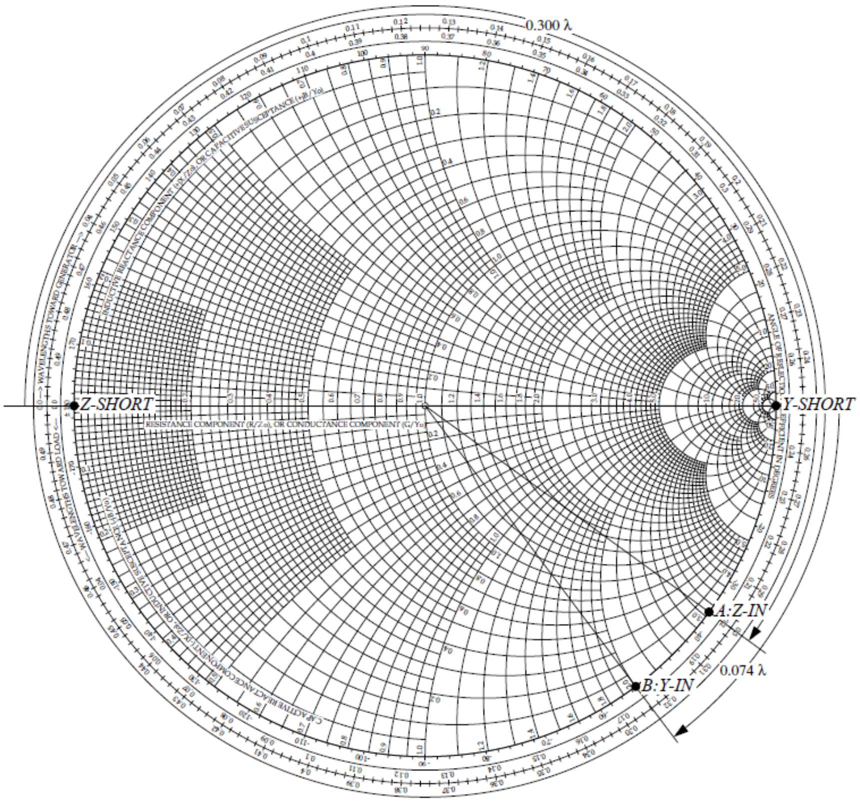


Figure P2.55: Solution of Problem 2.55.

Problem 2.66 A 200-Ω transmission line is to be matched to a computer terminal with $Z_L = (50 - j25)$ Ω by inserting an appropriate reactance in parallel with the line. If $f = 800$ MHz and $\varepsilon_r = 4$, determine the location nearest to the load at which inserting:

(a) A capacitor can achieve the required matching, and the value of the capacitor.

(b) An inductor can achieve the required matching, and the value of the inductor.

Solution:

(a) After entering the specified values for Z_L and Z_0 into Module 2.6, we have z_L represented by the red dot in Fig. P2.66(a), and y_L represented by the blue dot. By moving the cursor a distance $d = 0.093\lambda$, the blue dot arrives at the intersection point between the SWR circle and the $S = 1$ circle. At that point

$$y(d) = 1.026126 - j1.5402026.$$

To cancel the imaginary part, we need to add a reactive element whose admittance is positive, such as a capacitor. That is:

$$\begin{aligned}\omega C &= (1.54206) \times Y_0 \\ &= \frac{1.54206}{Z_0} = \frac{1.54206}{200} = 7.71 \times 10^{-3},\end{aligned}$$

which leads to

$$C = \frac{7.71 \times 10^{-3}}{2\pi \times 8 \times 10^8} = 1.53 \times 10^{-12} \text{ F}.$$

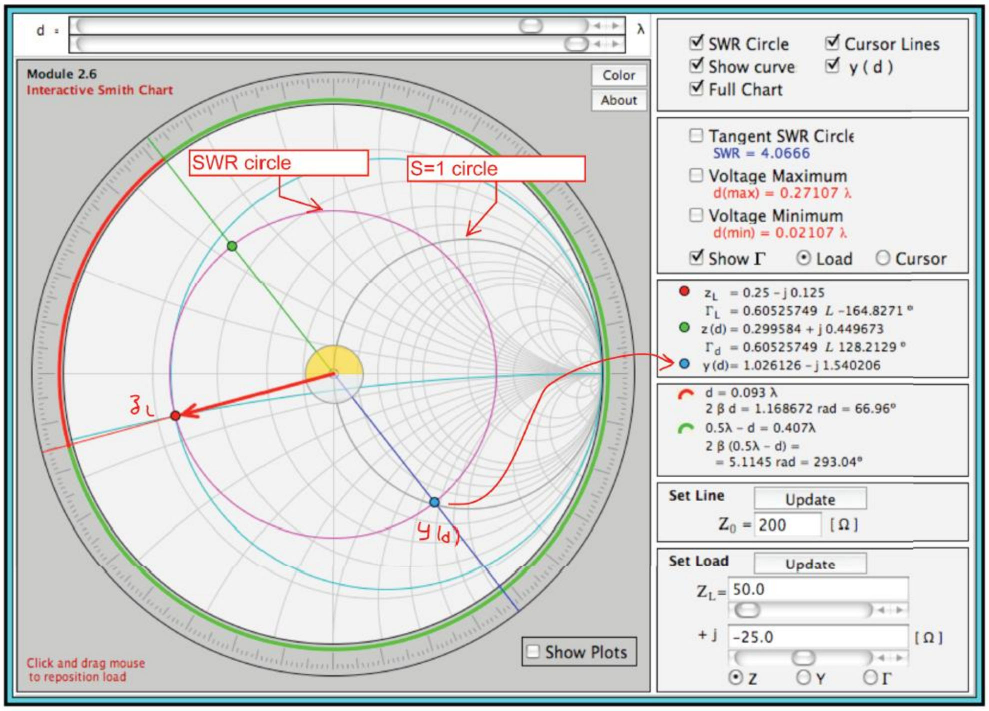


Figure P2.66(a)

(b) Repeating the procedure for the second intersection point [Fig. P2.66(b)] leads to

$$y(d) = 1.000001 + j1.520691,$$

at $d_2 = 0.447806\lambda$.

To cancel the imaginary part, we add an inductor in parallel such that

$$\frac{1}{\omega L} = \frac{1.520691}{200},$$

from which we obtain

$$L = \frac{200}{1.52 \times 2\pi \times 8 \times 10^8} = 2.618 \times 10^{-8} \text{ H}.$$

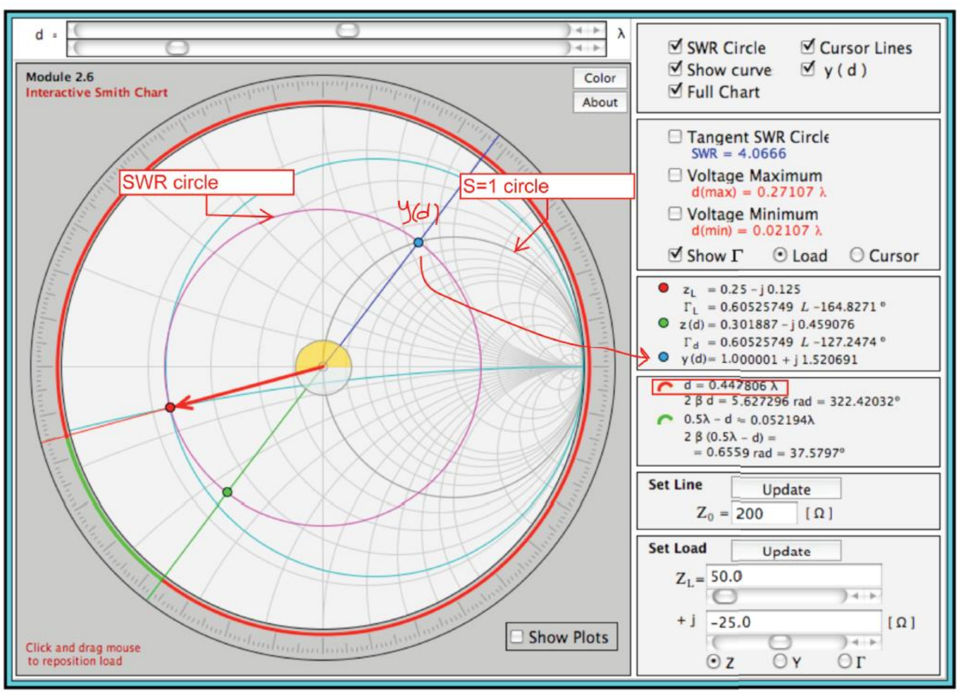


Figure P2.66(b)

Problem 2.68 A 50-Ω lossless line is to be matched to an antenna with $Z_L = (75 - j20)\ \Omega$ using a shorted stub. Use the Smith chart to determine the stub length and distance between the antenna and stub.

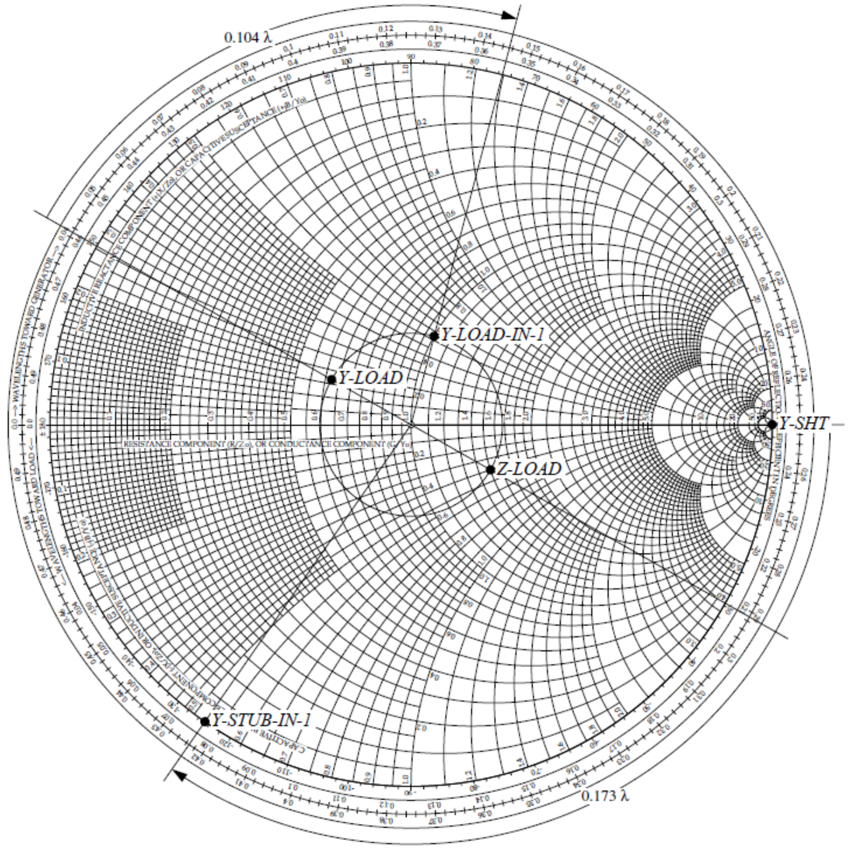


Figure P2.68: (a) First solution to Problem 2.68.

Solution: Refer to Fig. P2.68(a) and Fig. P2.68(b), which represent two different solutions.

$$z_L = \frac{Z_L}{Z_0} = \frac{(75 - j20)\ \Omega}{50\ \Omega} = 1.5 - j0.4$$

and is located at point *Z-LOAD* in both figures. Since it is advantageous to work in admittance coordinates, y_L is plotted as point *Y-LOAD* in both figures. *Y-LOAD* is at 0.041λ on the WTG scale.

For the first solution in Fig. P2.68(a), point *Y-LOAD-IN*-1 represents the point at which $g = 1$ on the SWR circle of the load. *Y-LOAD-IN*-1 is at 0.145λ on the WTG scale, so the stub should be located at $0.145\lambda - 0.041\lambda = 0.104\lambda$ from the load (or some multiple of a half wavelength further). At *Y-LOAD-IN*-1, $b = 0.52$, so a stub with an input admittance of $y_{\text{stub}} = 0 - j0.52$ is required. This point is *Y-STUB-IN*-1 and is at 0.423λ on the WTG scale. The short circuit admittance is denoted by point *Y-SHT*, located at 0.250λ. Therefore, the short stub must be $0.423\lambda - 0.250\lambda = 0.173\lambda$ long (or some multiple of a half wavelength longer).

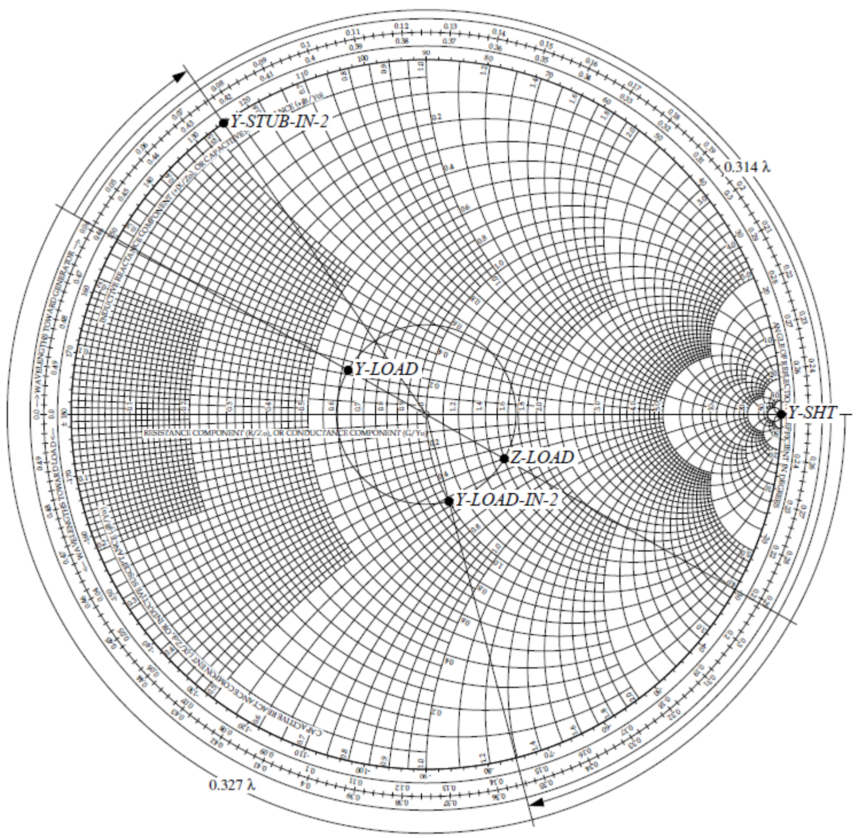


Figure P2.68: (b) Second solution to Problem 2.68.

For the second solution in Fig. P2.68(b), point *Y-LOAD-IN*-2 represents the point at which $g = 1$ on the SWR circle of the load. *Y-LOAD-IN*-2 is at 0.355λ on the WTG scale, so the stub should be located at $0.355\lambda - 0.041\lambda = 0.314\lambda$ from the load (or some multiple of a half wavelength further). At *Y-LOAD-IN*-2, $b = -0.52$, so a stub with an input admittance of $y_{\text{stub}} = 0 + j0.52$ is required. This point is *Y-STUB-IN*-2 and is at 0.077λ on the WTG scale. The short circuit admittance is denoted by point *Y-SHT*, located at 0.250λ. Therefore, the short stub must be $0.077\lambda - 0.250\lambda + 0.500\lambda = 0.327\lambda$ long (or some multiple of a half wavelength longer).

Problem 2.82 In response to a step voltage, the voltage waveform shown in Fig. P2.82 was observed at the midpoint of a lossless transmission line with $Z_0 = 50\ \Omega$ and $u_p = 2 \times 10^8$ m/s. Determine: (a) the length of the line, (b) Z_L, (c) R_g, and (d) V_g.

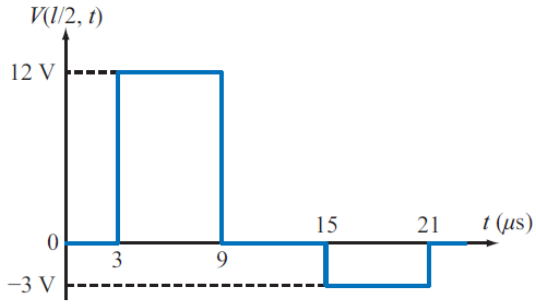


Figure P2.82: Circuit for Problem 2.82.

Solution:

(a) Since it takes 3 μs to reach the middle of the line, the line length must be

$$l = 2(3 \times 10^{-6} \times u_p) = 2 \times 3 \times 10^{-6} \times 2 \times 10^8 = 1200 \text{ m}.$$

(b) From the voltage waveform shown in the figure, the duration of the first rectangle is 6 μs, representing the time it takes the incident voltage V_1^+ to travel from the midpoint of the line to the load and back. The fact that the voltage drops to zero at $t = 9\ \mu$s implies that the reflected wave is exactly equal to V_1^+ in magnitude, but opposite in polarity. That is,

$$V_1^- = -V_1^+.$$

This in turn implies that $\Gamma_L = -1$, which means that the load is a short circuit:

$$Z_L = 0.$$

(c) After V_1^- arrives at the generator end, it encounters a reflection coefficient Γ_g. The voltage at 15 μs is composed of:

$$\begin{aligned} V &= V_1^+ + V_1^- + V_2^+ \\ &= (1 + \Gamma_L + \Gamma_L \Gamma_g) V_1^+ \\ \frac{V}{V_1^+} &= 1 - 1 - \Gamma_g \end{aligned}$$

From the figure, $V/V_1^+ = -3/12 = -1/4$. Hence,

$$\Gamma_g = \frac{1}{4},$$

which means that

$$R_g = \left(\frac{1 + \Gamma_g}{1 - \Gamma_g}\right) Z_0 = \left(\frac{1 + 0.25}{1 - 0.25}\right) 50 = 83.3\ \Omega.$$

(d)

$$\begin{aligned} V_1^+ &= 12 = \frac{V_g Z_0}{R_g + Z_0} \\ V_g &= \frac{12(R_g + Z_0)}{Z_0} = \frac{12(83.3 + 50)}{50} = 32 \text{ V}. \end{aligned}$$

Problem 3.5 Given vectors $\mathbf{A} = \hat{\mathbf{x}} + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}3$, $\mathbf{B} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}4$, and $\mathbf{C} = \hat{\mathbf{y}}2 - \hat{\mathbf{z}}4$, find

(a) A and $\hat{\mathbf{a}}$,

(b) the component of $\mathbf{B}$ along $\mathbf{C}$,

(c) θ_{AC},

(d) $\mathbf{A} \times \mathbf{C}$,

(e) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$,

(f) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$,

(g) $\hat{\mathbf{x}} \times \mathbf{B}$, and

(h) $(\mathbf{A} \times \hat{\mathbf{y}}) \cdot \hat{\mathbf{z}}$.

Solution:

(a) From Eq. (3.4),

$$A = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14},$$

and, from Eq. (3.5),

$$\hat{\mathbf{a}}_A = \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}3}{\sqrt{14}}.$$

(b) The component of $\mathbf{B}$ along $\mathbf{C}$ (see Section 3-1.4) is given by

$$B\cos\theta_{BC} = \frac{\mathbf{B} \cdot \mathbf{C}}{C} = \frac{-8}{\sqrt{20}} = -1.8\,.$$

(c) From Eq. (3.18),

$$\theta_{AC} = \cos^{-1}\frac{\mathbf{A} \cdot \mathbf{C}}{AC} = \cos^{-1}\frac{4+12}{\sqrt{14}\sqrt{20}} = \cos^{-1}\frac{16}{\sqrt{280}} = 17.0^\circ.$$

(d) From Eq. (3.27),

$$\mathbf{A} \times \mathbf{C} = \hat{\mathbf{x}}(2(-4) - (-3)2) + \hat{\mathbf{y}}((-3)0 - 1(-4)) + \hat{\mathbf{z}}(1(2) - 2(0)) = -\hat{\mathbf{x}}2 + \hat{\mathbf{y}}4 + \hat{\mathbf{z}}2.$$

(e) From Eq. (3.27) and Eq. (3.21),

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{A} \cdot (\hat{\mathbf{x}}16 + \hat{\mathbf{y}}8 + \hat{\mathbf{z}}4) = 1(16) + 2(8) + (-3)4 = 20.$$

Eq. (3.30) could also have been used in the solution. Also, Eq. (3.29) could be used in conjunction with the result of part (d).

(f) By repeated application of Eq. (3.27),

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{A} \times (\hat{\mathbf{x}}16 + \hat{\mathbf{y}}8 + \hat{\mathbf{z}}4) = \hat{\mathbf{x}}32 - \hat{\mathbf{y}}52 - \hat{\mathbf{z}}24.$$

Eq. (3.33) could also have been used.

(g) From Eq. (3.27),

$$\hat{\mathbf{x}} \times \mathbf{B} = -\hat{\mathbf{z}}4.$$

(h) From Eq. (3.27) and Eq. (3.21),

$$(\mathbf{A} \times \hat{\mathbf{y}}) \cdot \hat{\mathbf{z}} = (\hat{\mathbf{x}}3 + \hat{\mathbf{z}}) \cdot \hat{\mathbf{z}} = 1.$$

Eq. (3.29) and Eq. (3.25) could also have been used in the solution.

Problem 3.15 A certain plane is described by

$$2x+3y+4z=16.$$

Find the unit vector normal to the surface in the direction away from the origin.

Solution: Procedure:

1. Use the equation for the given plane to find three points, P_1, P_2 and P_3 on the plane.
2. Find vector $\mathbf{A}$ from P_1 to P_2 and vector $\mathbf{B}$ from P_1 to P_3.
3. Cross product of $\mathbf{A}$ and $\mathbf{B}$ gives a vector $\mathbf{C}$ orthogonal to $\mathbf{A}$ and $\mathbf{B}$, and hence to the plane.
4. Check direction of $\hat{\mathbf{c}}$.

Steps:

1. Choose the following three points:

$$\begin{aligned} &P_1 \text{ at } (0,0,4), \\ &P_2 \text{ at } (8,0,0), \\ &P_3 \text{ at } (0,\tfrac{16}{3},0). \end{aligned}$$

2. Vector $\mathbf{A}$ from P_1 to P_2

$$\mathbf{A}=\hat{\mathbf{x}}(8-0)+\hat{\mathbf{y}}(0-0)+\hat{\mathbf{z}}(0-4)=\hat{\mathbf{x}}8-\hat{\mathbf{z}}4$$

Vector $\mathbf{B}$ from P_1 to P_3

$$\mathbf{B}=\hat{\mathbf{x}}(0-0)+\hat{\mathbf{y}}\left(\frac{16}{3}-0\right)+\hat{\mathbf{z}}(0-4)=\hat{\mathbf{y}}\frac{16}{3}-\hat{\mathbf{z}}4$$

3.

$$\begin{aligned} \mathbf{C} &= \mathbf{A}\times\mathbf{B} \\ &=\hat{\mathbf{x}}(A_yB_z-A_zB_y)+\hat{\mathbf{y}}(A_zB_x-A_xB_z)+\hat{\mathbf{z}}(A_xB_y-A_yB_x) \\ &=\hat{\mathbf{x}}\left(0\cdot(-4)-(-4)\cdot\frac{16}{3}\right)+\hat{\mathbf{y}}((-4)\cdot 0-8\cdot(-4))+\hat{\mathbf{z}}\left(8\cdot\frac{16}{3}-0\cdot 0\right) \\ &=\hat{\mathbf{x}}\frac{64}{3}+\hat{\mathbf{y}}32+\hat{\mathbf{z}}\frac{128}{3} \end{aligned}$$

Verify that $\mathbf{C}$ is orthogonal to $\mathbf{A}$ and $\mathbf{B}$

$$\begin{aligned} \mathbf{A}\cdot\mathbf{C} &= \left(8\cdot\frac{64}{3}\right)+(32\cdot 0)+\left(\frac{128}{3}\cdot(-4)\right)=\frac{512}{3}-\frac{512}{3}=0 \\ \mathbf{B}\cdot\mathbf{C} &= \left(0\cdot\frac{64}{3}\right)+\left(32\cdot\frac{16}{3}\right)+\left(\frac{128}{3}\cdot(-4)\right)=\frac{512}{3}-\frac{512}{3}=0 \end{aligned}$$

4. $\mathbf{C}=\hat{\mathbf{x}}\frac{64}{3}+\hat{\mathbf{y}}32+\hat{\mathbf{z}}\frac{128}{3}$

$$\hat{\mathbf{c}}=\frac{\mathbf{C}}{|\mathbf{C}|}=\frac{\hat{\mathbf{x}}\frac{64}{3}+\hat{\mathbf{y}}32+\hat{\mathbf{z}}\frac{128}{3}}{\sqrt{\left(\frac{64}{3}\right)^2+32^2+\left(\frac{128}{3}\right)^2}}=\hat{\mathbf{x}}0.37+\hat{\mathbf{y}}0.56+\hat{\mathbf{z}}0.74.$$

$\hat{\mathbf{c}}$ points away from the origin as desired.

Problem 3.19 Vector field **E** is given by

$$\mathbf{E} = \hat{\mathbf{R}}\, 5R\cos\theta - \hat{\boldsymbol{\theta}}\frac{12}{R}\sin\theta\cos\phi + \hat{\boldsymbol{\phi}}3\sin\phi.$$

Determine the component of **E** tangential to the spherical surface $R = 2$ at point $P = (2, 30°, 60°)$.

Solution: At P, **E** is given by

$$\begin{aligned}\mathbf{E} &= \hat{\mathbf{R}}\, 5\times 2\cos 30° - \hat{\boldsymbol{\theta}}\frac{12}{2}\sin 30°\cos 60° + \hat{\boldsymbol{\phi}}3\sin 60° \\ &= \hat{\mathbf{R}}\, 8.67 - \hat{\boldsymbol{\theta}}1.5 + \hat{\boldsymbol{\phi}}2.6.\end{aligned}$$

The $\hat{\mathbf{R}}$ component is normal to the spherical surface while the other two are tangential. Hence,

$$\mathbf{E}_t = -\hat{\boldsymbol{\theta}}1.5 + \hat{\boldsymbol{\phi}}2.6.$$

Problem 3.25 Use the appropriate expression for the differential surface area $d\mathbf{s}$ to determine the area of each of the following surfaces:

(a) $r = 3;\ 0 \le \phi \le \pi/3;\ -2 \le z \le 2,$
(b) $2 \le r \le 5;\ \pi/2 \le \phi \le \pi;\ z = 0,$
(c) $2 \le r \le 5;\ \phi = \pi/4;\ -2 \le z \le 2,$
(d) $R = 2;\ 0 \le \theta \le \pi/3;\ 0 \le \phi \le \pi,$
(e) $0 \le R \le 5;\ \theta = \pi/3;\ 0 \le \phi \le 2\pi.$

Also sketch the outlines of each of the surfaces.

Solution:

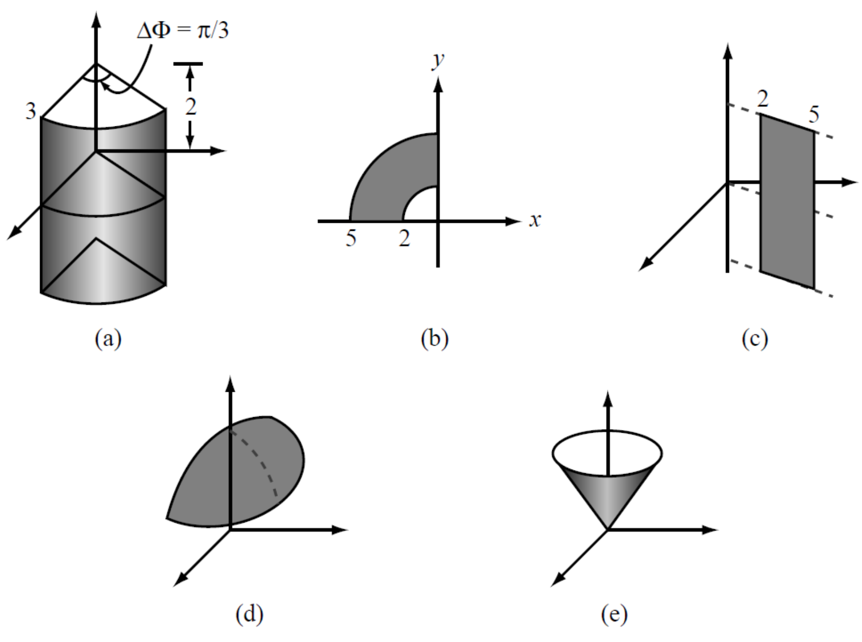


Figure P3.25: Surfaces described by Problem 3.25.

(a) Using Eq. (3.43a),

$$A = \int_{z=-2}^{2}\int_{\phi=0}^{\pi/3} (r)\big|_{r=3}\, d\phi\, dz = \left(\left(3\phi z\right)\Big|_{\phi=0}^{\pi/3}\right)\Big|_{z=-2}^{2} = 4\pi.$$

(b) Using Eq. (3.43c),

$$A = \int_{r=2}^{5}\int_{\phi=\pi/2}^{\pi} (r)\big|_{z=0}\, d\phi\, dr = \left(\left(\tfrac{1}{2}r^2\phi\right)\Big|_{r=2}^{5}\right)\Big|_{\phi=\pi/2}^{\pi} = \frac{21\pi}{4}.$$

(c) Using Eq. (3.43b),

$$A = \int_{z=-2}^{2} \int_{r=2}^{5} (1)\big|_{\phi=\pi/4}\, dr\, dz = \left((rz)\big|_{z=-2}^{2} \right)\Big|_{r=2}^{5} = 12.$$

(d) Using Eq. (3.50b),

$$A = \int_{\theta=0}^{\pi/3} \int_{\phi=0}^{\pi} \left(R^2 \sin\theta\right)\big|_{R=2}\, d\phi\, d\theta = \left((-4\phi \cos\theta)\big|_{\theta=0}^{\pi/3} \right)\Big|_{\phi=0}^{\pi} = 2\pi.$$

(e) Using Eq. (3.50c),

$$A = \int_{R=0}^{5} \int_{\phi=0}^{2\pi} (R\sin\theta)\big|_{\theta=\pi/3}\, d\phi\, dR = \left(\left(\tfrac{1}{2}R^2\phi \sin\frac{\pi}{3}\right)\Big|_{\phi=0}^{2\pi} \right)\Bigg|_{R=0}^{5} = \frac{25\sqrt{3}\pi}{2}.$$

Problem 3.36 Find the gradient of the following scalar functions:

(a) $T = 3/(x^2 + z^2)$,
(b) $V = xy^2z^4$,
(c) $U = z\cos\phi/(1 + r^2)$,
(d) $W = e^{-R}\sin\theta$,
(e) $S = 4x^2e^{-z} + y^3$,
(f) $N = r^2\cos^2\phi$,
(g) $M = R\cos\theta\sin\phi$.

Solution:

(a) From Eq. (3.72),

$$\nabla T = -\hat{\mathbf{x}}\frac{6x}{(x^2+z^2)^2} - \hat{\mathbf{z}}\frac{6z}{(x^2+z^2)^2}.$$

(b) From Eq. (3.72),

$$\nabla V = \hat{\mathbf{x}}y^2z^4 + \hat{\mathbf{y}}2xyz^4 + \hat{\mathbf{z}}4xy^2z^3.$$

(c) From Eq. (3.82),

$$\nabla U = -\hat{\mathbf{r}}\frac{2rz\cos\phi}{(1+r^2)^2} - \hat{\boldsymbol{\phi}}\frac{z\sin\phi}{r(1+r^2)} + \hat{\mathbf{z}}\frac{\cos\phi}{1+r^2}.$$

(d) From Eq. (3.83),

$$\nabla W = -\hat{\mathbf{R}}e^{-R}\sin\theta + \hat{\boldsymbol{\theta}}(e^{-R}/R)\cos\theta.$$

(e) From Eq. (3.72),

$$S = 4x^2e^{-z} + y^3,$$
$$\nabla S = \hat{\mathbf{x}}\frac{\partial S}{\partial x} + \hat{\mathbf{y}}\frac{\partial S}{\partial y} + \hat{\mathbf{z}}\frac{\partial S}{\partial z} = \hat{\mathbf{x}}8xe^{-z} + \hat{\mathbf{y}}3y^2 - \hat{\mathbf{z}}4x^2e^{-z}.$$

(f) From Eq. (3.82),

$$N = r^2\cos^2\phi,$$
$$\nabla N = \hat{\mathbf{r}}\frac{\partial N}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial N}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial N}{\partial z} = \hat{\mathbf{r}}2r\cos^2\phi - \hat{\boldsymbol{\phi}}2r\sin\phi\cos\phi.$$

(g) From Eq. (3.83),

$$M = R\cos\theta\sin\phi,$$
$$\nabla M = \hat{\mathbf{R}}\frac{\partial M}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial M}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial M}{\partial \phi} = \hat{\mathbf{R}}\cos\theta\sin\phi - \hat{\boldsymbol{\theta}}\sin\theta\sin\phi + \hat{\boldsymbol{\phi}}\frac{\cos\phi}{\tan\theta}.$$

Problem 3.44 Each of the following vector fields is displayed in Fig. P3.44 in the form of a vector representation. Determine $\nabla \cdot \mathbf{A}$ analytically and then compare the result with your expectations on the basis of the displayed pattern.

(a) $\mathbf{A} = -\hat{\mathbf{x}} \cos x \sin y + \hat{\mathbf{y}} \sin x \cos y$, for $-\pi \leq x, y \leq \pi$

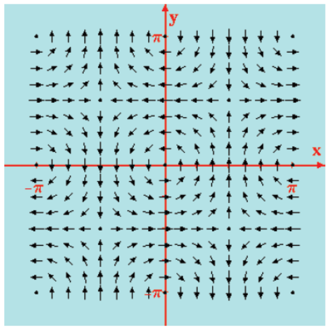


Figure P3.44(a)

Solution:

$$\mathbf{A} = -\hat{\mathbf{x}} \cos x \sin y + \hat{\mathbf{y}} \sin x \cos y$$

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \\ &= \frac{\partial}{\partial x}(-\cos x \sin y) + \frac{\partial}{\partial y}(-\sin x \cos y) \\ &= \sin x \sin y - \sin x \sin y = 0\end{aligned}$$

Yes, $\mathbf{A}$ is divergenceless everywhere.

(b) $\mathbf{A} = -\hat{\mathbf{x}} \sin 2y + \hat{\mathbf{y}} \cos 2x$, for $-\pi \leq x, y \leq \pi$

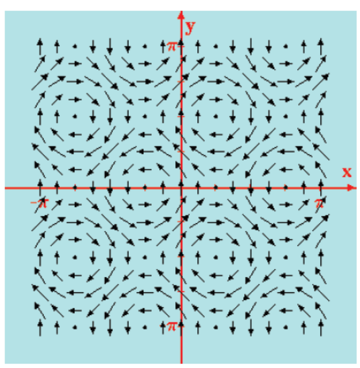


Figure P3.44(b)

Solution:

$$\mathbf{A} = -\hat{\mathbf{x}} \sin 2y + \hat{\mathbf{y}} \cos 2x$$

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \\ &= \frac{\partial}{\partial x}(-\sin 2y) + \frac{\partial}{\partial y}(\cos 2x) = 0\end{aligned}$$

Yes, $\mathbf{A}$ is divergenceless everywhere.

(c) $\mathbf{A} = -\hat{\mathbf{x}}xy + \hat{\mathbf{y}}y^2$, for $-10 \le x, y \le 10$

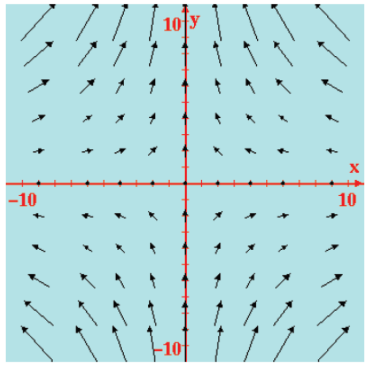


Figure P3.44(c)

Solution:

$$
\begin{aligned}
\mathbf{A} &= -\hat{\mathbf{x}}xy + \hat{\mathbf{y}}y^2 \\
\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \\
&= \frac{\partial}{\partial x}(-xy) + \frac{\partial}{\partial y}(y^2) = -y + 2y = y
\end{aligned}
$$

NO, $\mathbf{A}$ is not divergenceless everywhere. It is divergenceless only at $y = 0$.

(d) $\mathbf{A} = -\hat{\mathbf{x}}\cos x + \hat{\mathbf{y}}\sin y$, for $-\pi \le x, y \le \pi$

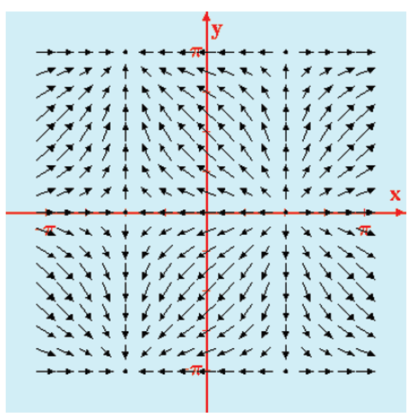


Figure P3.44(d)

Solution:

$$
\begin{aligned}
\mathbf{A} &= -\hat{\mathbf{x}}\cos x + \hat{\mathbf{y}}\sin y \\
\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \\
&= \frac{\partial}{\partial x}(-\cos x) + \frac{\partial}{\partial y}(\sin y) = \sin x + \cos y
\end{aligned}
$$

NO, $\mathbf{A}$ is not divergenceless everywhere.

(e) $\mathbf{A} = \hat{\mathbf{x}}x$, for $-10 \le x \le 10$

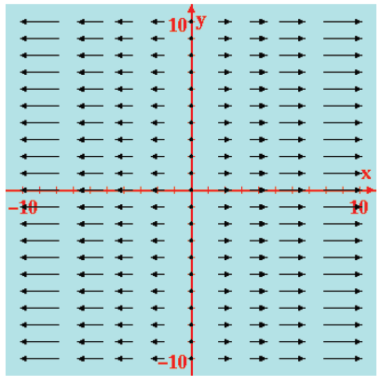


Figure P3.44(e)

Solution:

$$\mathbf{A} = \hat{\mathbf{x}}x$$
$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$
$$= 1$$

This indicates that the divergence of **A** is the same at all points in the defined space. In other words, every small volume is a source of flux (more flux leaving the volume than entering it), and the net generated flux is the same at all locations.

(f) $\mathbf{A} = \hat{\mathbf{x}}xy^2$, for $-10 \le x, y \le 10$

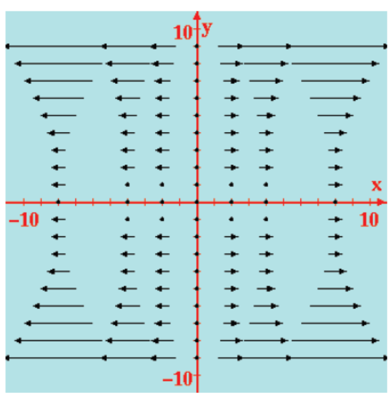


Figure P3.44(f)

Solution:

$$\mathbf{A} = \hat{\mathbf{x}}xy^2$$
$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$
$$= y^2$$

(g) $\mathbf{A} = \hat{\mathbf{x}}xy^2 + \hat{\mathbf{y}}x^2y$, for $-10 \le x, y \le 10$

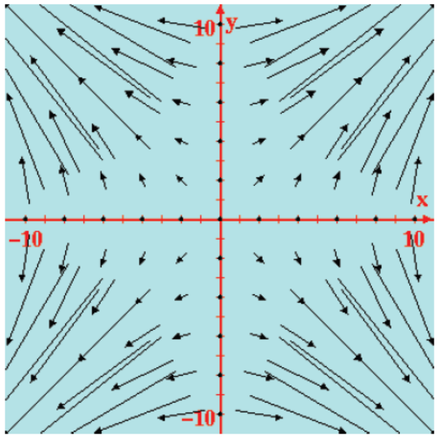


Figure P3.44(g)

Solution:

$$\mathbf{A} = \hat{\mathbf{x}}xy^2 + \hat{\mathbf{y}}x^2y$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= y^2 + x^2$$

(h) $\mathbf{A} = \hat{\mathbf{x}}\sin\left(\frac{\pi x}{10}\right) + \hat{\mathbf{y}}\sin\left(\frac{\pi y}{10}\right)$, for $-10 \le x, y \le 10$

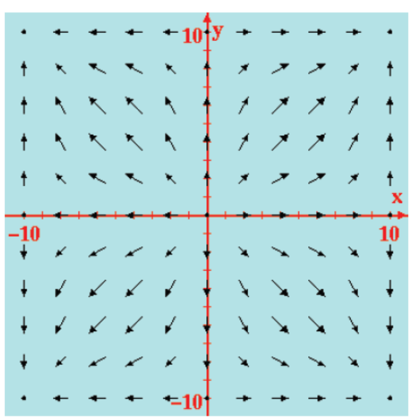


Figure P3.44(h)

Solution:

$$\mathbf{A} = \hat{\mathbf{x}}\sin(\pi x/10) + \hat{\mathbf{y}}\sin(\pi y/10)$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \frac{\pi}{10}[\cos(\pi x/10) + \cos(\pi y/10)]$$

(i) $\mathbf{A} = \hat{\mathbf{r}}r + \hat{\boldsymbol{\phi}}r\cos\phi$, for $\begin{cases} 0 \le r \le 10 \\ 0 \le \phi \le 2\pi. \end{cases}$

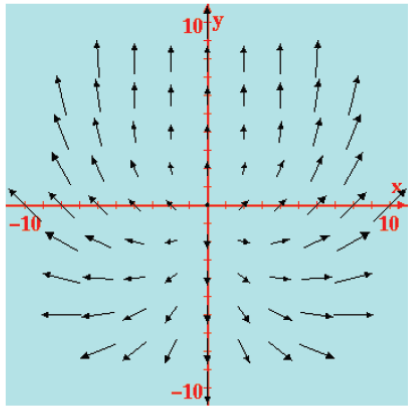


Figure P3.44(i)

Solution:

$$\mathbf{A} = \hat{\mathbf{r}}r + \hat{\boldsymbol{\phi}}r\cos\phi$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$= 2 - \sin\phi$$

(j) $\mathbf{A} = \hat{\mathbf{r}}r^2 + \hat{\boldsymbol{\phi}}r^2\sin\phi$, for $\begin{cases} 0 \le r \le 10 \\ 0 \le \phi \le 2\pi. \end{cases}$

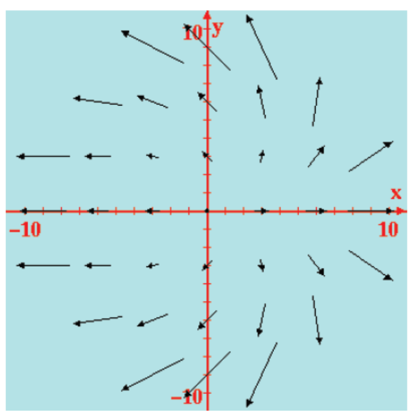


Figure P3.44(j)

Solution:

$$\mathbf{A} = \hat{\mathbf{r}}r^2 + \hat{\boldsymbol{\phi}}r^2\sin\phi$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$= 3r + r\cos\phi$$

Problem 3.46 For the vector field $\mathbf{E} = \hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy$, verify the divergence theorem by computing:

(a) the total outward flux flowing through the surface of a cube centered at the origin and with sides equal to 2 units each and parallel to the Cartesian axes, and

(b) the integral of $\nabla \cdot \mathbf{E}$ over the cube's volume.

Solution:

(a) For a cube, the closed surface integral has 6 sides:

$$\oint \mathbf{E} \cdot d\mathbf{s} = F_{\text{top}} + F_{\text{bottom}} + F_{\text{right}} + F_{\text{left}} + F_{\text{front}} + F_{\text{back}},$$

$$F_{\text{top}} = \int_{x=-1}^{1} \int_{y=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{z=1} \cdot (\hat{\mathbf{z}}\,dy\,dx)$$

$$= -\int_{x=-1}^{1} \int_{y=-1}^{1} xy\,dy\,dx = \left(\left(\frac{x^2y^2}{4}\right)\Big|_{y=-1}^{1}\right)\Bigg|_{x=-1}^{1} = 0,$$

$$F_{\text{bottom}} = \int_{x=-1}^{1} \int_{y=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{z=-1} \cdot (-\hat{\mathbf{z}}\,dy\,dx)$$

$$= \int_{x=-1}^{1} \int_{y=-1}^{1} xy\,dy\,dx = \left(\left(\frac{x^2y^2}{4}\right)\Big|_{y=-1}^{1}\right)\Bigg|_{x=-1}^{1} = 0,$$

$$F_{\text{right}} = \int_{x=-1}^{1} \int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{y=1} \cdot (\hat{\mathbf{y}}\,dz\,dx)$$

$$= -\int_{x=-1}^{1} \int_{z=-1}^{1} z^2\,dz\,dx = -\left(\left(\frac{xz^3}{3}\right)\Big|_{z=-1}^{1}\right)\Bigg|_{x=-1}^{1} = \frac{-4}{3},$$

$$F_{\text{left}} = \int_{x=-1}^{1} \int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{y=-1} \cdot (-\hat{\mathbf{y}}\,dz\,dx)$$

$$= -\int_{x=-1}^{1} \int_{z=-1}^{1} z^2\,dz\,dx = -\left(\left(\frac{xz^3}{3}\right)\Big|_{z=-1}^{1}\right)\Bigg|_{x=-1}^{1} = \frac{-4}{3},$$

$$F_{\text{front}} = \int_{y=-1}^{1} \int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{x=1} \cdot (\hat{\mathbf{x}}\,dz\,dy)$$

$$= \int_{y=-1}^{1} \int_{z=-1}^{1} z\,dz\,dy = \left(\left(\frac{yz^2}{2}\right)\Big|_{z=-1}^{1}\right)\Bigg|_{y=-1}^{1} = 0,$$

$$F_{\text{back}} = \int_{y=-1}^{1} \int_{z=-1}^{1} \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right)\Big|_{x=-1} \cdot (-\hat{\mathbf{x}}\,dz\,dy)$$

$$= \int_{y=-1}^{1} \int_{z=-1}^{1} z\,dz\,dy = \left(\left(\frac{yz^2}{2}\right)\Big|_{z=-1}^{1}\right)\Bigg|_{y=-1}^{1} = 0,$$

$$\oint \mathbf{E} \cdot d\mathbf{s} = 0 + 0 + \frac{-4}{3} + \frac{-4}{3} + 0 + 0 = \frac{-8}{3}.$$

(b)

$$\iiint \nabla \cdot \mathbf{E}\,dv = \int_{x=-1}^{1} \int_{y=-1}^{1} \int_{z=-1}^{1} \nabla \cdot \left(\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy\right) dz\,dy\,dx$$

$$= \int_{x=-1}^{1} \int_{y=-1}^{1} \int_{z=-1}^{1} (z - z^2)\,dz\,dy\,dx$$

$$= \left(\left(\left(xy\left(\frac{z^2}{2} - \frac{z^3}{3}\right)\right)\Big|_{z=-1}^{1}\right)\Bigg|_{y=-1}^{1}\right)\Bigg|_{x=-1}^{1} = \frac{-8}{3}.$$

Problem 3.52 Verify Stokes's theorem for the vector field $\mathbf{B} = (\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$ by evaluating:

(a) $\oint_C \mathbf{B}\cdot d\mathbf{l}$ over the semicircular contour shown in Fig. P3.52(a), and

(b) $\int_S (\nabla\times\mathbf{B})\cdot d\mathbf{s}$ over the surface of the semicircle.

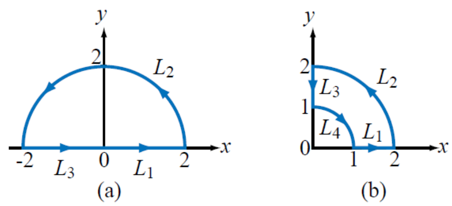


Figure P3.52: Contour paths for (a) Problem 3.52 and (b) Problem 3.53.

Solution:

(a)

$$\oint \mathbf{B}\cdot d\mathbf{l} = \int_{L_1} \mathbf{B}\cdot d\mathbf{l} + \int_{L_2} \mathbf{B}\cdot d\mathbf{l} + \int_{L_3} \mathbf{B}\cdot d\mathbf{l},$$

$$\mathbf{B}\cdot d\mathbf{l} = (\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot(\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}} r\,d\phi + \hat{\mathbf{z}}\,dz) = r\cos\phi\,dr + r\sin\phi\,d\phi,$$

$$\int_{L_1} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=0}^{2} r\cos\phi\,dr\right)\bigg|_{\phi=0,\ z=0} + \left(\int_{\phi=0}^{0} r\sin\phi\,d\phi\right)\bigg|_{z=0}$$

$$= \left(\tfrac{1}{2}r^2\right)\Big|_{r=0}^{2} + 0 = 2,$$

$$\int_{L_2} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{2} r\cos\phi\,dr\right)\bigg|_{z=0} + \left(\int_{\phi=0}^{\pi} r\sin\phi\,d\phi\right)\bigg|_{r=2,\ z=0}$$

$$= 0 + (-2\cos\phi)\big|_{\phi=0}^{\pi} = 4,$$

$$\int_{L_3} \mathbf{B}\cdot d\mathbf{l} = \left(\int_{r=2}^{0} r\cos\phi\,dr\right)\bigg|_{\phi=\pi, z=0} + \left(\int_{\phi=\pi}^{\pi} r\sin\phi\,d\phi\right)\bigg|_{z=0}$$

$$= \left(-\tfrac{1}{2}r^2\right)\Big|_{r=2}^{0} + 0 = 2,$$

$$\oint \mathbf{B}\cdot d\mathbf{l} = 2 + 4 + 2 = 8.$$

(b)

$$\nabla\times\mathbf{B} = \nabla\times(\hat{\mathbf{r}} r\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)$$

$$= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0 - \frac{\partial}{\partial z}(\sin\phi)\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}(r\cos\phi) - \frac{\partial}{\partial r}0\right)$$

$$+ \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(r(\sin\phi)) - \frac{\partial}{\partial\phi}(r\cos\phi)\right)$$

$$= \hat{\mathbf{r}}0 + \hat{\boldsymbol{\phi}}0 + \hat{\mathbf{z}}\frac{1}{r}(\sin\phi + (r\sin\phi)) = \hat{\mathbf{z}}\sin\phi\left(1 + \frac{1}{r}\right),$$

$$\iint \nabla\times\mathbf{B}\cdot d\mathbf{s} = \int_{\phi=0}^{\pi}\int_{r=0}^{2}\left(\hat{\mathbf{z}}\sin\phi\left(1+\frac{1}{r}\right)\right)\cdot(\hat{\mathbf{z}} r\,dr\,d\phi)$$

$$= \int_{\phi=0}^{\pi}\int_{r=0}^{2} \sin\phi\,(r+1)\,dr\,d\phi = \left(\left(-\cos\phi\left(\tfrac{1}{2}r^2 + r\right)\right)\Big|_{r=0}^{2}\right)\bigg|_{\phi=0}^{\pi} = 8.$$